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CRITICISMS AND DISCUSSIONS.

PANDIAGONAL MAGICS OF ORDERS 6 AND 10 WITH MINIMAL NUMBERS.

In *The Monist,* Vol. XXII, p. 304, appeared an article by Messrs. Andrews and Frierson on magic squares of orders $\equiv 2$ (mod. 4) based on information transmitted by me, as referred to in the second and other paragraphs of their paper.

It may possibly be inferred from this article that squares obtained by the methods described contain the lowest groups of numbers with which pandiagonal squares of the respective orders can be constructed. Also in a following paper by Mr. Sayles, *loc. cit.,* p. 472, this suggestion is apparently taken for granted. It was not my intention, however, that any such inference should be drawn. It is true that the groups of numbers chosen are the lowest with which such squares can be constructed by the path method or by any of the allied forms of procedure; but I have long been aware

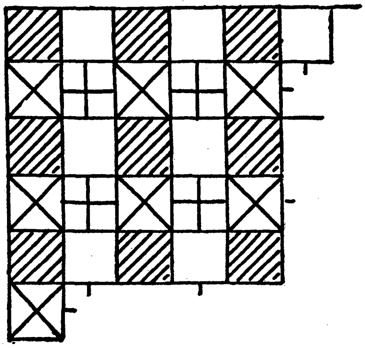

Fig. 1.

that such squares may be made with much lower groups, only in these cases the squares will be irregular, non-La Hireian, and must be approached by special tactic.

It is known that magic squares of order $\equiv 2 \pmod{4}$ made with consecutive numbers cannot be pandiagonal; but I believe that no rigorous proof of the fact has yet been published. The following is quite satisfactory.

39	37	4	10	13	17
35	34	7	12	14	18
32	29	9	15	16	19
21	24	25	31	11	8
22	26	28	33	6	5
23	27	30	36	3	1

Fig. 2.

Take a square lattice of any even order $2m$ and make a double chequer of its cells as shown in Fig. 1, where there are four species of cells symmetrically disposed. Suppose this lattice filled with the first $(2m)^2$ natural numbers so that it is magic on all rows, columns, and diagonals. The mean number is $(4m^2+1)/2$ and the magic sum is $m(4m^2+1)=\Sigma$. Let the sum of the numbers in all the shaded cells $=A$, in all the blank cells $=B$, and in all the cells with diagonals drawn $=C$. Then by summing all the numbers in

39	36	30	21	23	32
35	26	24	27	31	34
29	25	22	28	33	37
3	7	12	18	15	11
6	9	13	16	14	5
8	17	19	10	4	1

Fig. 3.

39	17	13	10	4	37
35	7	12	18	14	34
29	9	19	16	15	32
8	25	24	21	31	11
6	26	22	28	33	5
3	36	30	27	23	1

Fig. 4.

the 1st, 3d, 5th, etc., rows, we have $A+B=m\Sigma$. Similarly by summing the 1st, 3d, etc., columns $A+C=m\Sigma$, and from the alternate $/$diagonals $B+C=m\Sigma$. These three equations yield $A=B=C=m\Sigma/2=2m^4+m^2/2$. The assumed square cannot therefore exist when m is odd, for then $m^2/2$ would be fractional. It will be noticed that we have only made use of the $/$ diagonals, so that a square of order $\equiv 2$ (mod. 4) cannot even be 1/2 pandiagonal using consecutive numbers. The same result also follows for associated squares, for if an associated square of these orders existed it could be transformed into a pandiagonal by the $A-D$ method.

If m is odd, what is the lowest group of integers with which the square of order $2m$ can be magic and associated? Clearly unity must be one of the group, for otherwise by diminishing every number in the finished magic by unity, we could secure the result with a lower group of numbers. Let t be the highest integer in the group; then the mean number is $(t+1)/2$, and the magic sum $\Sigma=m(t+1)$. Since the square is associated it can always be transformed to a pandiagonal and from the above investigation we have $A=B=C=m\Sigma/2=m^2(t+1)/2$. Since m is odd, $t+1$ must be even and therefore t odd, say $t=2x+1$. If we now write down the complementary couplets and their half differences, we have:

1	$2x+1$	x
2	$2x$	$x-1$
3	$2x-1$	$x-2$
.	.	.
.	.	.
.	.	.
.	.	.
.	.	.
$x-1$	$x+3$	2
x	$x+2$	1
$x+1$		

and evidently if the square is to be associated, the middle number of the group $x+1$ must always be omitted. By the method of Complementary Differences[1] we have to form m equations using $2m^2$ of the half differences, each once and once only. It follows that the sum of the half differences used must be even, for the m

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[1] See *The Monist*, Vol. XX, p. 434.

equations separate the $2m^2$ numbers into m sets, the sum of each set being even. But the sum of all the half differences from 1 to x is $x(x+1)/2$, and this is odd when $x \equiv 1$ or 2 (mod. 4), but even when $x \equiv 0$ or 3 (mod. 4). In the former case the sum of the $x-2m^2$ half differences which have to be omitted must be odd, and in the latter case their sum must be even.

Consider the simplest case, $m=3$, i. e., order 6. If $t=37$, $x=18 \equiv 2$ (mod. 4) and the sum of the half differences is therefore odd, and since no half difference can now be omitted, the construction is impossible. If $t=39$, then $x=19 \equiv 3$ (mod. 4) and the one half difference to be omitted must be even. Thus we have to omit the mean number 20, and the one complementary couplet corresponding to any one of the 9 even half differences from 2 to 18 inclusive. The necessary equations can always be found, as the following table shows, so that there are exactly nine ways in which we can choose 36 from among the first 39 natural numbers to form an associated magic square of order 6.

		HALF DIFERENCES
1	39	19
2	38	18
3	37	17
4	36	16
5	35	15
6	34	14
7	33	13
8	32	12
9	31	11
10	30	10
11	29	9
12	28	8
13	27	7
14	26	6
15	25	5
16	24	4
17	23	3
18	22	2
19	21	1
20		

From these equations the nine associated squares may be obtained, and each of them can be transformed to a pandiagonal square by the A-D method. The magic sum is always 120, but the last case (half difference 18 omitted) is the most interesting because it contains only two numbers > 36; all the others containing three such numbers, this case will be used in illustration. Fig. 2 gives the distribution of the numbers in rows, Fig. 3 that in columns,

39	17	13	37	4	10
35	7	12	34	14	18
29	9	19	32	15	16
3	36	30	1	23	27
6	26	22	5	33	28
8	25	24	11	31	21

Fig. 5.

38	16	12	36	3	9
34	6	11	33	13	17
28	8	18	31	14	15
2	35	29	0	22	26
5	25	21	4	32	27
7	24	23	10	30	20

Fig. 6. $\Sigma = 114$.

1	4	9	10	11	101	99	98	96	91
7	16	18	19	20	92	90	89	87	82
21	26	27	28	30	81	80	79	75	73
32	35	37	39	40	71	70	68	66	62
41	45	47	48	51	61	60	58	55	54
50	49	46	44	43	53	56	57	59	63
42	38	36	34	33	64	65	67	69	72
31	29	25	24	23	74	76	77	78	83
22	17	15	14	12	84	85	86	88	97
13	8	6	5	3	93	94	95	100	103

Fig. 7.

1	7	12	3	6	49	43	46	45	47
16	9	19	22	14	38	36	41	32	37
27	30	21	24	29	31	25	35	26	28
39	40	23	33	42	15	17	34	20	18
51	48	94	44	50	8	5	13	11	4
100	93	91	99	96	54	60	10	56	53
86	84	70	87	89	62	71	81	64	65
76	78	69	79	73	75	80	83	74	77
67	72	63	68	66	90	82	85	95	88
57	59	58	61	55	98	101	92	97	103

Fig. 8.

1	9	91	99	96	98	101	10	11	4
16	7	19	87	89	90	82	92	20	18
27	30	21	79	73	75	80	81	26	28
39	40	70	68	66	62	71	35	32	37
51	48	58	61	55	54	60	41	45	47
57	59	63	44	50	49	43	46	56	53
67	72	69	33	42	38	36	34	64	65
76	78	23	24	29	31	25	83	74	77
86	84	12	22	14	15	17	85	97	88
100	93	94	3	6	8	5	13	95	103

Fig. 9. $\Sigma = 520$.

and by combining these two we get the associated magic Fig. 4. Applying the A – D method to Fig. 4 gives Fig. 5, a pandiagonal magic. This last square answers a question that I have repeatedly proposed for years past, but without receiving reply, for by decreasing all its numbers by unity we obtain Fig. 6, which is a pandiagonal magic with zero allowed and only one number greater than 36.

Turning to the next double of odd order 10, we have $m = 5$. If $t = 101$, then $x = 50 \equiv 2 \pmod{4}$ and the construction is again impossible. In fact, if $t = 4m^2 + 1$, then $x = 2m^2$, and since m is odd x is always $\equiv 2$, and no associated magic of order $n = 2 \pmod{4}$ can be constructed with numbers not exceeding $n^2 + 1$. If $t = 103$, $x = 51 \equiv 3 \pmod{4}$ and we have to omit one even half difference from the series 2, 4,....50. No doubt the 25 sets of equations all exist. The set for the case when the half difference 50 is omitted is exemplified in the annexed table of equations. From these we get Fig. 7 for rows and Fig. 8 for columns. Fig. 9 is an associated magic of order 10 with no number higher than 103, and Fig. 10 is the resulting pandiagonal magic. By diminishing each number

1	9	91	99	96	4	11	10	101	98
16	7	19	87	89	18	20	92	82	90
27	30	21	79	73	28	26	81	80	75
39	40	70	68	66	37	32	35	71	62
51	48	58	61	55	47	45	41	60	54
100	93	94	3	6	103	95	13	5	8
86	84	12	22	14	88	97	85	17	15
76	78	23	24	29	77	74	83	25	31
67	72	69	33	42	65	64	34	36	38
57	59	63	44	50	53	56	46	43	49

Fig. 10. $\Sigma = 520$.

in Fig. 10 by unity we get a pandiagonal magic of order 10 with zero allowed and only one number > 100.

If we choose higher values of t, the number of selections is largely increased. In the case of order 6, if we go up to $t = 41$, we have $x = 20 \equiv 0 \pmod{4}$ and the sum of the two half differences to be omitted must be even: that is, both must be even or both odd. We can select them in ninety different ways, but in one sense these will not all be fundamentally new selections, because in the nine cases where we take 20 as one of a pair of even half differences, we exclude the numbers 41 from the finished square so that the contained numbers extend from 2 to 40, and if we then diminish all of them by unity, we get the nine cases already enumerated when $t = 39$. Indeed if we stipulate that unity must always be included in the final square the enumeration for any value of t includes that for all lower values. Thus if $t = 49$, then $x = 24 \equiv 0 \pmod{4}$ and the six half differences excluded must have an even sum, so that they must include an even number of odd half differences, viz., 0, 2, 4, or 6. The enumeration is therefore $2(12\,C0 \times 12\,C6 + 12\,C2 \times 12\,C4) = 67{,}188$, which is the number of different ways in which 36 suitable numbers can be selected from the first 49 on the understanding that unity must always be included.

4	0	34	5	30	35
10	28	29	11	3	27
14	23	20	12	18	21
31	6	1	32	36	2
25	33	9	26	8	7
24	18	15	22	13	16

Fig. 11.

It happens by chance that in all the cases so far illustrated $x \equiv 0$ or $3 \pmod{4}$, so that the sum of the omitted half differences has always to be even; the first case of the opposite kind with order 6 occurs when $t = 43$. We have then $x = 21 \equiv 1 \pmod{4}$ and the sum of the omitted half differences must be odd, that is, all three odd, or two even and one odd. The following equations suffice when the half differences 20, 19, 18 are omitted.

$21+15+11=17+16+14$	$21+8+1=15+13+2$
$13+8+7=12+10+6$	$14+11=9+7+6+3$
$9+2+1=5+4+3$	$17+10+5=16+12+4$

TABLE OF EQUATIONS FOR PANDIAGONAL MAGICS OF ORDER 6 WITH MINIMAL NUMBERS.

HALF DIFF'S OMITTED	NUMBERS OMITTED	EQUATIONS FOR ROWS	EQUATIONS FOR COLUMNS
2	18, 20, 22	$19+18=17+10+9+1$ $16+15=13+11+4+3$ $14+12=8+7+6+5$	$19+16+12=18+15+14$ $17+8+3=11+10+7$ $13+5+1=9+6+4$
4	16, 20, 24	$19+13=14+9+6+3$ $18+17=15+11+7+2$ $16+10=12+8+5+1$	$19+18+1=17+13+8$ $16+7+3=14+10+2$ $15+9+5=12+11+6$
6	14, 20, 26	$19+12=11+9+7+4$ $18+17=15+14+5+1$ $16+10=13+8+3+2$	$19+17+10=18+16+12$ $15+11+8=14+13+7$ $9+2+1=5+4+3$
8	12, 20, 28	$19+16=15+10+6+4$ $18+17=14+13+5+3$ $12+9=11+7+2+1$	$19+18+7=17+16+11$ $15+3+2=13+6+1$ $12+10+5=14+9+4$
10	10, 20, 30	$19+12=14+9+5+3$ $18+16=15+11+7+1$ $17+8=13+6+4+2$	$19+18+8=17+16+12$ $15+14+4=13+11+9$ $7+3+2=6+5+1$
12	8, 20, 32	$19+16=15+10+6+4$ $18+17=14+13+5+3$ $11+8=9+7+2+1$	$19+17+7=18+16+9$ $15+3+2=13+6+1$ $14+8+4=11+10+5$
14	6, 20, 34	$19+17=15+10+8+3$ $18+13=16+7+6+2$ $12+9=11+5+4+1$	$19+13+12=18+17+9$ $15+7+1=16+4+3$ $10+6+5=11+8+2$
16	4, 20, 36	$18+10=12+8+7+1$ $17+13=19+6+3+2$ $15+14=11+9+5+4$	$19+10+1=15+9+6$ $18+8+2=14+11+3$ $17+7+5=13+12+4$
18	2, 20, 38	$19+17=16+10+7+3$ $15+14=13+8+6+2$ $12+9=11+5+4+1$	$19+15+9=17+14+12$ $16+6+5=13+11+3$ $10+4+2=8+7+1$

TABLE OF EQUATIONS FOR A PANDIAGONAL MAGIC OF ORDER 10 WITH MINIMAL NUMBERS, WHEN THE HALF DIFERENCE IS OMITTED.

$$\begin{aligned}
51+48+43+42+41&=49+47+46+44+39\\
45+36+34+33+32&=40+38+37+35+30\\
31+26+25+24+22&=29+28+27+23+21\\
20+17+15+13+12&=19+18+16+14+10\\
11+7+5+4+1&=9+8+6+3+2
\end{aligned}$$

$$\begin{aligned}
51+36+25+13+1&=48+34+24+15+5\\
45+43+22+12+4&=41+32+26+20+7\\
40+33+31+29&=42+39+18+17+11+6\\
49+30+28+19+8&=47+35+27+16+9\\
46+38+23+10+1&=44+37+21+14+3
\end{aligned}$$

By the foregoing methods the reader should have no difficulty in constructing a pandiagonal magic square of any double odd order n without using any number higher than n^2 if the following two concessions are granted, (I) zero allowed, (II) one duplicate permitted, that is, one number is allowed to appear twice. Fig. 11 is a pandiagonal magic made under the above conditions, 18 being the duplicate number and no number > 36. $\Sigma = 108$.

C. Planck.